Divisors

Wednesday, November 13, 2024 9:11 AM

Today X stands for an integral (reduced, irreducible) noetherian scheme.

Def: · The group of Weil divisors is the additive free abelian subgp spanned by the irreducible closed subsets of codim 1. It's denoted $\mathrm{Weil}(X)$.

· A member of the group of Weil divisors is called a Weil divisor. So a Weil divisor is a formal sum $\sum_{i=1}^{t} n_i Z_i$, where Z_i irr and $\mathrm{codim}_X(Z_i) = 1$, $n_i \in \mathbb{Z}$

· $\sum_{i=1}^{t} n_i Z_i + \sum_{i=1}^{t} n_i' Z_i = \sum_{i=1}^{t} (n_i + n_i') Z_i$

Eg: In $\mathbb{P}^1_{\mathbb{C}}$, codim 1 pts are closed pts. So a Weil divisor looks like $\sum_i n_i([x_i : y_i])$, such as $[1:0] - [0:1]$.

Eg. In $\mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x,y])$ $\quad V(x) - 3V(y)$ is a Weil div.

Eg. Take the rational function $x/y^3 \in \mathbb{C}(x,y)$ on $\mathbb{A}^2_{\mathbb{C}}$. $V(x) - 3y(y)$ reads the zeroes and poles of this rational function with order.

For today unless otherwise mentioned we shall assume X is regular in codimension 1. This means for every $\xi \in X$ such that $\dim \mathcal{O}_{X,\xi} = 1$, $\mathcal{O}_{X,\xi}$ is a regular ring.

If Z is irr of codim 1 and η_Z be the generic point of Z (i.e $\overline{\{\eta_Z\}} = Z$), $\dim \mathcal{O}_{Z,\eta_Z} = 1$.
Indeed, recall for $x \in X$, $\mathrm{codim}_X(\overline{\{x\}}) = \dim \mathcal{O}_{X,x}$.
The codim of a point $x \in X$ is by definition $\dim \mathcal{O}_{X,x} = \mathrm{codim}_X(\overline{\{x\}})$.

§: Divisor of a rational function.

Recall: $K(X)$ = Field of rational function of X

$$= \mathcal{O}_{X,\eta} \text{ where } \eta \text{ is the generic pt of } X.$$

$$= \{ f \in \mathcal{O}_X(U) \mid U \underset{\text{open}}{\subseteq} X \} \Big/ \begin{array}{l} f \in \mathcal{O}_X(U) \\ \sim \tilde{f} \in \mathcal{O}_X(\tilde{U}) \\ \text{iff } \exists V \subseteq U \cap \tilde{U} \\ \text{s.t } f|_V = \tilde{f}|_V \end{array}$$

Let X be regular in codim 1. Given an irr closed $Z \subseteq X$ of codim 1, we want to define $\mathrm{ord}_Z : K(X) \to \mathbb{Z} \cup \{\infty\}$ Let η_Z be the generic pt of Z. The local ring $\mathcal{O}_{X,\eta_Z}$ is reg of dim 1.

For $0 \neq f \in \mathcal{O}_{X,\eta_Z}$, define $\mathrm{ord}_Z(f) = \max\{n \mid f \in m^n_{\eta_Z} \mathcal{O}_{X,\eta_Z}\} < \infty$.
Since $\mathcal{O}_{X,\eta_Z}$ is reg of dim 1, $m_{\eta_Z} = (\varphi)$. So $\mathrm{ord}_Z(f)$ is the unique non. negative integer n_0 s.t $f = u \cdot \varphi^{n_0}$ for some unit u in $\mathcal{O}_{X,\eta_Z}$. Note $\mathrm{ord}_Z(fg) = \mathrm{ord}_Z f + \mathrm{ord}_Z g$.

Now extend $\mathrm{ord}_Z : \mathcal{O}_{X,\eta_Z} \setminus \{0\} \to \mathbb{Z}$ to $K(X) = \mathrm{Frac}(\mathcal{O}_{X,\eta_Z})$ by

$-\infty$.

Now extend $\mathrm{ord}_Z : \mathcal{O}_{X,\eta_Z} \setminus \{0\} \to \mathbb{Z}$ to $K(X) = \mathrm{Frac}(\mathcal{O}_{X,\eta_Z})$ by setting:

① $\mathrm{ord}_Z(0) = \infty$.

② For $0 \neq h \in K(X)$, write $h = f/g$.

Define $\mathrm{ord}_Z(f/g) = \mathrm{ord}_Z(f) - \mathrm{ord}_Z(g)$.

Rmk: For $f \in K(X)$, $\mathrm{ord}_Z(f)$ measures the order of zeroes (or poles) of f at the generic point of Z or along Z.

eg: $X = \mathbb{A}^2_{\mathbb{C}}$, $f = x^3/y$, $Z_1 = V(x)$, $Z_2 = V(y)$

$\mathrm{ord}_{Z_1} f = 3$, $\mathrm{ord}_{Z_2} f = -1$. For any other codim 1 subset $Z \subseteq \mathbb{A}^2_{\mathbb{C}}$, $\mathrm{ord}_Z(f) = 0$

Prop / Def. For $f \in K(X)^*$,

$$\sum_{\substack{Z \text{ irr of} \\ \text{codim } 1}} \mathrm{ord}_Z(f) \cdot Z \quad \text{is an Weil divisor.}$$

This is called the divisor of f and denoted $\mathrm{div}(f)$.

Pf. Since X is noeth, choose a finite affine open cover $X = \bigcup_{i=1}^{n} U_i$.

For each i, write $f = f_i/g_i$. For an irr closed subset Z of codim 1 such that $Z \cap U_i \neq \emptyset$ or equivalently s.t. the gen pt of Z, $\eta_Z \in U_i$,

$\mathrm{ord}_Z(f) \neq 0 \Rightarrow$ either $\mathrm{ord}_Z(f_i) \neq 0$ or $\mathrm{ord}_Z(g_i) \neq 0$.

Now for $h \in \mathcal{O}_X(U_i)$, and a codim 1 pt $p \in \mathrm{Spec}(\mathcal{O}_X(U_i)) = U_i$, $\mathrm{ord}_p(h) \neq 0$

$\iff h$ is not a unit in $\mathcal{O}_X(U_i)_p$

$\iff h \in p\,\mathcal{O}_X(U_i)_p$

$\iff h \in p$ in $\mathcal{O}_X(U_i)$

Now by Krull's Thm the set

$\{ q \in \mathrm{Spec}(\mathcal{O}_X(U_i)) \mid h \in q \}$ is finite since each such prime is a min prime of $\mathcal{O}_X(U_i)/h$.

So # of codim 1 pts ξ in $\mathcal{O}_X(U_i)$ s.t. $\mathrm{ord}_{\overline{\{\xi\}}}(f_i) \neq 0$ or $\mathrm{ord}_{\overline{\{\xi\}}}(g_i) \neq 0$ is finite.

For an irr closed $Z \subseteq X$, $\mathrm{ord}_Z(f) \neq 0$

$\Rightarrow \exists\, i \quad \mathrm{ord}_{Z \cap U_i}(f_i/g_i) \neq 0$ for some i

$\Rightarrow \sum \mathrm{ord}_Z(f)\, Z$ is a finite sum

Prop: • For any irr closed Z of codim 1,

$\ldots \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g)$.

Proof: • For any irr closed Z of codim 1,
$\mathrm{ord}_Z(fg) = \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g)$.

- Thus $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$.
- $\mathrm{div}(1) = 0$ • $\mathrm{div}(1/f) = -\mathrm{div}(f)$
- Thus $\{\mathrm{div}(f) \mid f \in K(X)^*\} \subseteq \mathrm{Weil}(X)$
  is a subgp.

Def: ① The quotient gp $\dfrac{\mathrm{Weil}(X)}{\{\mathrm{div}(f) \mid f \in k(X)^*\}}$ is
called the Weil divisor class gp or simply the divisor class gp and denoted $\mathrm{Cl}(X)$.

② $D_1, D_2 \in \mathrm{Weil}(X)$ are called linearly equivalent if $\exists f \in K(X)$,
s.t $D_1 - D_2 = \mathrm{div}(f)$.
$D_1 \sim D_2$ means D_1 and D_2 are linearly equivalent.

Eg. 1) $\mathrm{Cl}(\mathbb{A}^n_k) = \{0\}$ where k is a field.
$\mathbb{A}^n_k = \mathrm{Spec}(k[x_1, \dots, x_n])$. Z irr closed
of codim $1 \implies \exists$ an irr pol f s.t
$Z = V(f)$. Note $\mathrm{div}(f) = Z$.
So $K(\mathbb{A}^n_k) \xrightarrow{\mathrm{div}} \mathrm{Weil}(\mathbb{A}^n_k)$ is surj.

Eg 2: $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$. Given a codim 1 irr closed
Z, $\exists$ a homo irr pol f, such that
$Z = V(f)$, define $\deg Z = \deg f$.

Have a om. $\deg : \mathrm{Weil}(\mathbb{P}^n_k) \to \mathbb{Z}$. ($\deg(V(x_0)) = 1$)

Given $D \in \mathrm{Weil}(\mathbb{P}^n_k)$, write

$$D = \sum_{i=1}^{r} n_i V(f_i) - \sum_{j=1}^{m} m_j V(g_j) \text{ where}$$

$n_i, m_j > 0$.

$$\deg D = \sum_{i=1}^{r} n_i \deg(f_i) - \sum_{j=1}^{m} m_j \deg(g_j) = 0$$

$$\iff \deg\left(\prod_{i=1}^{r} f_i^{n_i}\right) = \deg\left(\prod_{j=1}^{m} g_j^{m_j}\right)$$

$$\iff h = \frac{\prod^{r} f_i^{n_i}}{\prod_{j=1} g_j^{m_j}} \in K(\mathbb{P}^n_k) \text{ and}$$

$$D = \mathrm{div}(h).$$

So $\deg$ induces an isom $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$.
$[V(x_0)]$ being a gen.

End of 19.11.25 lecture

Start of 21.11.25 Lecture

Def. Y integral scheme. $K(Y)$ is the constant
sheaf $K(Y)(U) = K(Y)$.

Sheaves associated to Weil divisors

For $D' = \sum n_i Z_i \in \mathrm{Div}(X)$,

- $D'|_U = \sum_{r} n_i (Z_i \cap U) \in \mathrm{Div}(U)$; • $D' \geq 0 \overset{\text{def}}{\iff} n_i \geq 0 \; \forall i$.

[illegible] $\mathrm{Weil}(X)$. define the sheaf $\mathcal{O}_X(D) \subseteq K(X)$ [illegible]

- $D'|_U = \sum_{r} n_i (Z_i \cap U) \in \mathrm{Div}(U)$; • $D \geq 0 \Leftrightarrow n_i \geq 0$ ..

Def. Given $D \in \mathrm{Weil}(X)$, define the sheaf $\mathcal{O}_X(D) \subseteq \underline{K(X)}$ by $\mathcal{O}_X(D)(U) = \{ f \in K(X) \mid (\mathrm{div}(f) + D)|_U \geq 0 \} \subseteq K(X)(U) = K(X)$

Prop. $\mathcal{O}_X(D) \subseteq \underline{K(X)}$ is an additive subgp sheaf

Pf. It is straightforward to check $\mathcal{O}_X(D)$ is a sheaf.

We verify $\mathcal{O}_X(D)(U) \subseteq K(X)$ is a subgp.

Let $f_1, f_2 \in \mathcal{O}_X(D)(U)$ and $Z \underset{\text{codim } 1}{\subseteq} X$ such that $Z \cap U \neq \emptyset$. Choose a generator of the maximal ideal $\mathcal{O}_{X,\eta_Z}$, denoted t.

$t^{\mathrm{ord}_Z(f_1)} \mid f_1$, $t^{\mathrm{ord}_Z(f_2)} \mid f_2$ in $\mathcal{O}_{X,\eta_Z}$.

Thus $t^{\min\{\mathrm{ord}_Z(f_1), \mathrm{ord}_Z(f_2)\}} \mid f_1 + f_2$

$\Rightarrow$ $\mathrm{ord}_Z(f_1+f_2) \geq \min\{\mathrm{ord}_Z(f_1), \mathrm{ord}_Z(f_2)\}$.

$\Rightarrow$ If $(D + \mathrm{div} f_1)|_U \geq 0$, $(D + \mathrm{div}(f_2))|_U \geq 0$. Then $(D + \mathrm{div}(f_1+f_2))|_U \geq 0 \Rightarrow f_1 + f_2 \in \mathcal{O}_X(U)$

Since $\mathrm{ord}_Z(-f_1) = \mathrm{ord}_Z f_1$, $-f_1 \in \mathcal{O}_X(U)$.

Thus $(\mathcal{O}_X(D)(U), +)$ is a group.

Rmk. Say $D = \sum_{i=1} n_i Z_i$, $\mathrm{div} f = \sum_{i=1} m_i Z_i$, $m_i = \mathrm{ord}_{Z_i}(f)$.

$f \in \mathcal{O}_X(D)(U) \Leftrightarrow$ for each i s.t $Z_i \cap U \neq \emptyset$. $n_i + \mathrm{ord}_{Z_i}(f) \geq 0$

$\Leftrightarrow$ if $n_i \geq 0$, f can have a pole of order atmost n_i along Z_i, If $n_i \leq 0$, $\mathrm{ord}_{Z_i}(f) \geq -n_i$ i.e f must have a zero of ord atleast n_i along Z_i

Prop. $\mathcal{O}_X(D) \in \mathrm{QCoh}(X)$.

Pf We prove that for any affine open $U \subseteq X$, $f \in \mathcal{O}_X(U)$

The $\underline{\text{inclusion}}$ $\mathcal{O}_X(D)(U)[1/f] \to \mathcal{O}_X(D)(U_f)$ where $U_f = \mathrm{Spec}(\mathcal{O}_X(U)[1/f]) \subseteq U$

is an isom of $\mathcal{O}_X(U)$-mods.

To prove surjectivity, take $\varphi \in \mathcal{O}_X(D)(U_f)$. Write $(\mathrm{div}\,\varphi + D)|_U = a_1 Z_1 + \cdots + a_r Z_r$

where $Z_i \subseteq U$ are prime divisors. Assume $Z_1, \ldots, Z_s \subseteq U \setminus U_f = V(f)$.

and $Z_j \cap U_f \neq \emptyset$ for $j \geq s+1$. As $(\mathrm{div}\,\varphi + D)|_{U_f} \geq 0$

This means $\mathrm{ord}_{Z_i}(f) > 0$ for $1 \leq i \leq r$.

$a_j \geq 0$ for $j \geq s+1$

Choose $n \in \mathbb{N}$ s.t $\mathrm{ord}_{Z_i}(f^n) > -a_i$ for $\forall\, 1 \leq i \leq s$.

note $(\mathrm{div}(\varphi f^n) + D)|_U = (\mathrm{div}\, f^n|_U + \mathrm{div}(\varphi + D)|_U) \geq 0$.

Choose $n \in \mathbb{N}$ s.t $\operatorname{ord}_{Z_i}(f^n) > -a_i$ for $\forall\, 1 \le i \le r$.
note $(\operatorname{div}(\varphi f^n) + D)_{|U} = (\operatorname{div} f^n_{|U} + \operatorname{div}(\varphi + D)_{|U}) \geq 0$.
Thus $\varphi f^n \in \mathcal{O}_X(D)(U)$. So φ is in the image of
$\mathcal{O}_X(D)(U)[1/f] \longrightarrow \mathcal{O}_X(D)(U_f)$.

__Prop.__ ① Let η be the generic pt of X. $\mathcal{O}_X(D)_\eta \xrightarrow{\sim} k(X)$.
② The multiplication map on $k(X)$ induces an $\mathcal{O}_X$-linear map $\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_X(D_1 + D_2)$
③ $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$ as $\mathcal{O}_X$-mods $\iff D_1 \sim D_2$

__Pf:__ ① Suppose $D = a_1 Z_1 + a_2 Z_2 + \dots + a_r Z_r$, Z_i prime divisors. $a_i \in \mathbb{Z}$.
Choose an affine open subset V of
$X - \bigcup_{i=1}^{r} Z_i$

$\mathcal{O}_X(D)(V) \supseteq \mathcal{O}_X(V)$ as $D_{|V} = 0$.
Thus $k(X) \supseteq \mathcal{O}_X(D)_\eta \supseteq k(X) \Rightarrow \mathcal{O}_X(D)_\eta = k(X)$

② Clear
③ $\Leftarrow$ Take $f \in k(X)$ s.t $D_1 - D_2 = \operatorname{div}(f)$
The multiplication by f map denoted $\lambda_f : \underline{k(X)} \longrightarrow \underline{k(X)}$
takes $\mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_X(D_1)$. The inverse is given by
$\lambda_{1/f}$.
$\Rightarrow$ Any isom $\mathcal{O}_X(D_1) \xrightarrow{\sim} \mathcal{O}_X(D_2)$ induces an isom
$k(X) = \mathcal{O}_X(D_1)_\eta \longrightarrow \mathcal{O}_X(D_2)_\eta = k(X)$ of $k(X)$-mods
Any $k(X)$-linear isom $k(X) \to k(X)$ is given by
λ_f for some $f \in k(X) - \{0\}$
So $\lambda_f(\mathcal{O}_X(D_1)) \subseteq \mathcal{O}_X(D_2)$ and $\lambda_{1/f}(\mathcal{O}_X(D_2)) \subseteq \mathcal{O}_X(D_1)$.
We claim $\operatorname{div} f = D_1 - D_2$. Indeed for a prime
divisor Z, suppose Z appears with multiplicity a_1, a_2 in D_1 and D_2
respectively. Suppose $m_{\eta_Z} = (t)$. We can choose an affine
nbhd $\operatorname{Spec}(A)$ of η_Z s.t ① $t \in A$ ② If q is the pt 1
prime ideal corresponding to η_Z, $q = (t)$
Thus $1/t^{a_1} \in \mathcal{O}_X(D_1)(\operatorname{Spec}(A))$. So $f/t^{a_1} \in \mathcal{O}_X(D_2)(\operatorname{Spec}(A))$
$\Rightarrow \operatorname{ord}_Z f - a_1 \geq -a_2 \Rightarrow \operatorname{ord}_Z(f) \geq a_1 - a_2$
A similar argument with $1/f$ shows
$\operatorname{ord}_Z(1/f) - a_2 \geq -a_1 \Rightarrow \operatorname{ord}_Z(f) \leq a_1 - a_2$
Thus $\operatorname{ord}_Z(f) = a_1 - a_2$ $\forall Z$ and we are done.

__Example.__ Recall That $\deg : \operatorname{Cl}(\mathbb{P}^n_k) \longrightarrow \mathbb{Z}$ is an isom; $\mathbb{P}^n = \operatorname{Proj}(k[x_0, \dots, x_n])$
Take a degree d divisor D on $\mathbb{P}^n_k$. We claim
$\mathcal{O}_{\mathbb{P}^n}(D) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^n}(d)$. Since $D \sim d V(x_0)$, W.L.O.G we take
$D = d V(x_0)$. Note $\mathcal{O}_{\mathbb{P}^n}(D)(D_+(x_i)) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^n}(D_+(x_i)) \dfrac{x_i^d}{x_0^d}$
$\mathcal{O}_{\mathbb{P}^n}(d)(D_+(x_i)) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^n}(D_+(x_i)) \cdot x_i^d$.

(The same description works even when $d < 0$).

Define $\mathcal{O}_{\mathbb{P}^n}(d)(D_+(x_i)) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(D)(D_+(x_i))$ by sending
$x_i^d \longmapsto x_i^d / x_0^d$
These glue (check) to given an isom $\mathcal{O}_{\mathbb{P}^n}(d) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^n}(d V(x_0))$.

[illegible] V be an m-eg sc

These glue (check) to give ...

Cartier divisors. Y be an m-eg sc

Def. • Y integral $\underline{K(Y)}^*$ is the constant sheaf of abelian gps $\underline{K(Y)}^*(U) = K(Y)^*$, under pdt.

• $\mathcal{O}_Y^*$ is the sub sheaf of abelian gps given by $\mathcal{O}_Y^*(U) =$ unit of the ring $\mathcal{O}_Y(U)$.

Def. The group (abelian) of Cartier divisors is the group $\Gamma(Y, \underline{K(Y)}^*/\mathcal{O}_Y^*) = \mathrm{Cart}(Y)$

• An elt of $\Gamma(Y, \underline{K(Y)}^*/\mathcal{O}_Y^*)$ is called a Cartier divisor.

• There is a natural ~~map~~ gp homo.

$$K(Y)^* \longrightarrow \mathrm{Cart}(Y)$$

The quotient gp, denoted $\mathrm{CaCl}(Y)$, is called the Cartier divisor class group.

Rmk. Given $s \in \mathrm{Cart}(Y)$, $\exists$ an open covering $Y = \bigcup_{i \in I} U_i$ and $f_i \in K(Y)^*$ s.t. $s|_U$ is represented by f_i. On $U_i \cap U_j$, $f_i = u_{ij} f_j$ ~~for~~ for some $u_{ij} \in \mathcal{O}_Y(U_i \cap U_j)^*$

* X noeth, integral, reg in codim 1.

Def/Prop. Given Z irr closed of codim 1,

$$(\underline{K(X)}^*/\mathcal{O}_X^*)_{\eta_Z} \cong \frac{\underline{K(X)}^*_{\eta_Z}}{(\mathcal{O}_X^*)_{\eta_Z}} = \frac{K(X)^*}{\mathcal{O}^*_{X,\eta_Z}}$$

For $h \in \mathcal{O}^*_{X,\eta_Z}$, $\mathrm{ord}_Z(h) = 0$.

Thus $\mathrm{ord}_Z : K(X)^* \longrightarrow \mathbb{Z}$ factors through $K(X)^*/\mathcal{O}^*_{X,\eta_Z}$.

• Given $s \in \mathrm{Cart}(X)$, define $\boxed{\mathrm{ord}_Z(s) = \mathrm{ord}_Z(s_{\eta_Z} \in \frac{K(X)^*}{\mathcal{O}^*_{X,\eta_Z}})}$

• Given $s \in \mathrm{Cart}(X)$, define

$$\boxed{\mathrm{div}(s) = \sum \mathrm{ord}_Z(s) \cdot Z \in \mathrm{Weil}(X).}$$

• The following claim gives a way to 'think about' and justifies why the sum in $\mathrm{div}(s)$ for $s \in \mathrm{Cart}(X)$ on the right hand is finite.

Claim: Given $s \in \mathrm{Cart}(X)$, choose a finite open covering $X = \bigcup_{i=1}^{r} U_i$ such that for each i, $\exists f_i \in K(X)^*$ satisfying $s|_{U_i} = f_i$ in $\underline{K(X)}^*/\mathcal{O}_X^*(U_i)$

... (f_i) ...

covering ... $i \in I$
satisfying $s|_{U_i} = f_i$ in $\underline{K(X)^*}/\mathcal{O}_X^*(U_i)$
Then $\mathrm{div}(s)|_{U_i} = \mathrm{div}(f_i)|_{U_i}$

Pf. For $Z \underset{\text{irr closed codim } 1}{\subseteq} X$ such that $Z \cap U_i \neq \emptyset$.
$\mathrm{ord}_Z(s) = \mathrm{ord}_Z(f_i)$. So the claim follows.

Prop. (i) $\mathrm{div} : \mathrm{Cart}(X) \to \mathrm{Weil}(X)$ is a gp homo.
(ii) For $f \in K(X)^*$, $\mathrm{div}(f \in \mathrm{Cart}(X)) = \mathrm{div}(f \in K(X)^*)$
So div induces a gp homo.
$\overline{\mathrm{div}} : \mathrm{CaCl}(X) := \dfrac{\mathrm{Cart}(X)}{K(X)^*} \longrightarrow \mathrm{Cl}(X)$
(iii) When X is normal, div and $\overline{\mathrm{div}}$ are injective.

We need the following Lemma.

Lemma: Let X be an integral normal scheme (in particular regular in codim 1);
$f \in K(X)$. Then (i) $f \in \mathcal{O}_X(X) \iff \mathrm{div}(f) \geq 0$.
(ii) If $\mathrm{div}(f) = 0$, f is a unit in $\mathcal{O}_X(X)$

Pf. Clearly (i) $\Rightarrow$ (ii)

(i) Fact. Let R be a normal ring. Then $R = \bigcap_{\mathrm{ht}(p)=1} R_p$, where
The intersection is taken in $\mathrm{Frac}(R)$.
$\Rightarrow$ is clear. For $\Leftarrow$ by working in an affine covering,
we can assume, $X \cong \mathrm{Spec}(R)$, for some normal R, W.L.O.G.
Then $\mathrm{div}(f) \geq 0 \Rightarrow f/1 \in R_p \ \forall$ ht 1 prime $p \in R$.
Thus $f \in R$ by the Fact.

Pf of Prop: (i), (ii) is clear. We only prove $\overline{\mathrm{div}}$ is injective,
The argument for div is similar.
Let $s \in \mathrm{Cart}(X)$ such that $\mathrm{div}(s) = \mathrm{div}(f) \in \mathrm{Weil}(X)$ for some $f \in K(X)$. Choose an affine covering $X = \bigcup_{i \in I} U_i$ s.t
$s|_{U_i} = [f_i]$ for some $f_i \in K(X)$.
So $\mathrm{div}(f)|_{U_i} = \mathrm{div}(f_i)|_{U_i}$ $\forall i$ $\Rightarrow$ $\mathrm{div}(f/f_i)|_{U_i} = 0 \Rightarrow f/f_i \in \mathcal{O}_X(U_i)^*$ $\forall i$
$\Rightarrow$ The constant section of $\underline{K(X)^*}/\mathcal{O}_X^*$ given by f agrees with s.

End of 21.11.25 Lecture

... divisor D is called locally principal

Def. • A Weil divisor D on X is called <u>locally principal</u> if $\exists$ a (finite) open covering $X = \cup U_i$ and $f_i \in K(X)^*$ such that $D|_{U_i} = \operatorname{div} f_i|_{U_i}$ for each i.

• A Weil divisor D is called <u>effective</u> if
$$D = \sum_{i=1}^{r} n_i Z_i \quad \text{where } n_i \geq 0$$

So we have $\operatorname{div} : \operatorname{Cart}(X) \to \operatorname{Weil}(X)$

Prop. Assume X is reg in codim 1.

1) $\operatorname{div}(\operatorname{Cart}(X)) \subseteq$ locally principal Weil divisors.

2) When X is normal, the above containment is an equality.

Pf. 1) follows from claim $*$.

2) Given a locally principal Weil divisor D, choose a finite open covering $X = \bigcup_{i=1}^{r} U_i$ and $f_i \in K(X)^*$ such that $D|_{U_i} = \operatorname{div} f_i|_{U_i}$. Since $\operatorname{div} f_i|_{U_i \cap U_j} = \operatorname{div}(f_j)|_{U_i \cap U_j}$
$= D|_{U_i \cap U_j}$

$\operatorname{div}(f_i/f_j)|_{U_i \cap U_j} = 0$. <u>Since X is normal</u>
$f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^*$. So $\{f_i \in \underline{K(X)^*}/\mathcal{O}_X^*(U_i)\}_i$ glue to give a global section
$$s \in \Gamma(X, \underline{K(X)^*}/\mathcal{O}_X^*).$$
By claim $*$, $\operatorname{div}(s) = D$.

• We examine surjectivity of $\operatorname{Cart}(X) \xrightarrow{\operatorname{div}} \operatorname{Weil}(X)$.

The following Thm is crucial.

Thm. Let A be a noetherian ring.
A is a unique factorization domain
$\iff$ A is normal and $\operatorname{Cl}(\operatorname{Spec} A) = \{0\}$.

Pf. See Prop 6.2, Hart.

Thm. Let X be a normal, noetherian scheme
$\operatorname{Cart}(X) \xrightarrow{\operatorname{div}} \operatorname{Weil}(X)$ is surjective (eq. bijective) for every $x \in X$, $\mathcal{O}_{X,x}$ is a
... factorial i.e. ...

Thm. Let ... $\mathrm{Cart}(X) \xrightarrow{div} \mathrm{Weil}(X)$ is surjective (eq. bijective)

$\iff$ X is <u>locally factorial</u>, i.e. for every $x \in X$, $\mathcal{O}_{X,x}$ is a U.F.D.

$\iff$ $\overline{div} : \mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Cl}(X)$.

Pf. We only verify $\Leftarrow$ of the top $\iff$.

We know $div(\mathrm{Cart}(X))$ = locally principal Weil divisors of X.

So it's enough to check that any prime divisor D is locally principal. Since $D|_{X \setminus D} = div(1)|_{X\setminus D}$, we need to show around every $x \in D$, D is locally principal. D corresponds to a ht 1 prime in $\mathcal{O}_{X,x}$, which is principal as $\mathcal{O}_{X,x}$ is a U.F.D. So $\exists$ an affine nbhd U of x s.t. The prime ideal corresponding to D in $\mathcal{O}_X(U)$ is also principal, say generated by f.

Then $D|_U = div f|_U$.

Thm. When X is regular, $div: \mathrm{Cart}(X) \xrightarrow{\sim} \mathrm{Weil}(X)$ and $\overline{div} : \mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Cl}(X)$.

Pf. Regular local rings are U.F.D.

§ <u>Picard group and CaCl</u>

The results of This section are true for any integral scheme, which are not necessarily locally noeth.

Thm. Y be an integral scheme. Given a $\mathcal{S} \in \mathrm{Cart}(Y)$, define

$$\mathcal{L}(\mathcal{S})(U) = \{ f \in K(Y)^* \mid f_x \mathcal{S}_x \in \mathcal{O}_{Y,x} - \{0\} / \mathcal{O}_{Y,x}^* \ \forall x \in U \} \cup \{0\}$$

$$\subseteq \underline{K(Y)}(U) = K(Y)$$

Then i) $\mathcal{L}(\mathcal{S})$ is a sub $\mathcal{O}_Y$-mod sheaf of The sheaf of abelian grps $(\underline{K(Y)}, +)$.

Then 1) $\mathcal{L}(\delta)$ is a sub $\mathcal{O}_Y$-mod sheaf of the sheaf of abelian gps $(\underline{K(Y)}, +)$.

2) $\mathcal{L}(\delta)$ is invertible

Pf 1) is straightforward.

2) Claim** Given $\delta \in \mathrm{Cart}(Y)$. Choose a covering

$$Y = \bigcup_{i \in I} U_i \quad \text{s.t} \quad \delta|_{U_i} = f_i \text{ in } \Gamma(U_i, K(Y)^\times / \mathcal{O}_Y^\times)$$

Then

$$\begin{array}{ccc} \mathcal{L}(\delta)|_{U_i} & \xleftarrow{\sim} & \mathcal{O}_{U_i} . \\ 1/f_i & \longleftarrow & 1 \end{array}$$

In particular $\mathcal{L}(\delta)(U_i) = 1/f_i \, \mathcal{O}_Y(U_i)$ and $\mathcal{L}(\delta)|_{U_i} \xrightarrow{\sim} 1/f_i \, \mathcal{O}_Y(U_i)$ when U_i is affine.

Pf $1/f_i \in \Gamma(U_i, \mathcal{L}(\delta))$ as $f_i \cdot 1/f_i \in \mathcal{O}^*_{Y,y}$ $\forall y \in U_i$

$$\begin{array}{ccc} \mathcal{L}(\delta)|_{U_i} & \xrightarrow{\cdot f_i} & \mathcal{O}_{U_i} \\ \cup & & \cup \\ \underline{K(Y)} & \xrightarrow{f_i} & \underline{K(Y)} \end{array}$$

gives the inverse.

Thm. Let Y be an integral scheme.

1) The map $\delta \longmapsto \mathcal{L}(\delta)$ gives a bijection

$$\mathrm{Cart}(Y) \longleftrightarrow \{\text{invertible subsheaves of } \underline{K(Y)}\}.$$

2) The map in (1) induces an isom of groups

$$\mathrm{CaCl}(Y) \xrightarrow{\sim} \{\text{isom classes of invertible subsheaves of } \underline{K(Y)}\}.$$

where the group structure on the right again comes from $\otimes \, \mathcal{O}_Y$

3) We have a commutative diag of isoms of gps.

$$\begin{array}{ccc} \mathrm{CaCl}(Y) & \xrightarrow[(2)]{\sim} & \{\text{isom classes of invertible subsheaves of } \underline{K(Y)}\} \\ & \searrow & \downarrow \\ & & \mathrm{Pic}(Y) \end{array}$$

$$\longrightarrow \mathrm{Pic}(Y)$$

Pf. 1) <u>Injection:</u> Assume $s_\alpha, s_\beta \in \mathrm{Cart}(Y)$ have the same image. Choose an open covering

$$Y = \bigcup_{i \in I} U_i$$

such that $\exists\ f_i^\alpha, f_i^\beta$ for each i satisfying, $s_\alpha|_{U_i} = f_i^\alpha$, $s_\beta|_{U_i} = f_i^\beta$ $\forall i$.

Thus $\frac{1}{f_i^\alpha} \mathcal{O}_Y(U_i) = \frac{1}{f_i^\beta} \mathcal{O}_Y(U_i)$ $\forall i$

$\Rightarrow$ f_i^β / f_i^α, $f_i^\alpha / f_i^\beta \in \mathcal{O}_Y(U_i)$ $\forall i$

$\Rightarrow$ $f_i^\beta / f_i^\alpha \in \mathcal{O}_Y(U_i)^*$ $\forall i$

$\Rightarrow$ $s_\alpha = s_\beta \in \Gamma(Y, \underline{K(Y)}^* / \mathcal{O}_Y^*)$.

<u>Surjectivity:</u> Given an invertible $\mathcal{O}_Y$ submod $\mathcal{F}$ of $\underline{K(Y)}$, choose a open covering $Y = \bigcup_{i \in I} U_i$ such that for each i $\mathcal{F}|_{U_i} = f_i \cdot \mathcal{O}_{U_i}$ for some $f_i \in K(Y)^*$

Since $f_i \cdot \mathcal{O}_{U_i}|_{U_i \cap U_j} = f_j \mathcal{O}_{U_j}|_{U_i \cap U_j}$

$f_i / f_j \in \mathcal{O}_{U_i}(U_i \cap U_j)^*$

$\Rightarrow$ $\{1/f_i \in \Gamma(U_i, \underline{K(Y)}^* / \mathcal{O}_Y^*)\}_{i \in I}$ glue to give a section $s \in \Gamma(Y, \underline{K(Y)}^* / \mathcal{O}_Y^*)$

By Claim $**$, $\mathcal{L}(s) = \mathcal{F}$.

2) By claim $**$, we have

$$\mathcal{L}(s_1) \otimes_{\mathcal{O}_Y} \mathcal{L}(s_2) \xrightarrow{\sim} \mathcal{L}(s_1 s_2)$$

So we only need to check: if $\mathcal{L}(s) \xrightarrow{\sim} \mathcal{O}_Y$, Then $\exists f \in K(Y)^*$ such that $s = f$ in $\Gamma(Y, \underline{K(Y)}^* / \mathcal{O}_Y^*)$.

To that end, fix an isom

$$\mathcal{O}_Y \xrightarrow{\sim} \mathcal{L}(s).$$

This isom

To that end, Γx ....

$$\mathcal{O}_Y \xrightarrow{\sim} \mathcal{L}(s).$$

Denote the image of $1 \in \mathcal{O}_Y(Y)$ via this isom by $g \in K(Y)^*$.

So $\mathcal{L}(s)(U) = g \cdot \mathcal{O}_Y(U) \quad \forall\, U \subseteq_{open} Y$.

Choose an open cover $Y = \bigcup_{i \in I} U_i$ s.t. $s|_{U_i} = f_i$ for some $f_i \in K(Y)^*$.

By Claim $**$, $\quad \mathcal{L}(s)(U_i) = \frac{1}{f_i} \mathcal{O}_Y(U_i)$

Thus $\quad \frac{1}{f_i} \mathcal{O}_Y(U_i) = g \cdot \mathcal{O}_Y(U_i) \quad \forall i \qquad \frac{1}{f_i} = g x$

$$\Rightarrow \quad g / \tfrac{1}{f_i} \in \mathcal{O}_Y(U_i)^* \quad \forall i \qquad g = \tfrac{1}{f_i} \cdot y$$

$$\Rightarrow \quad s = \frac{1}{g} \in \Gamma(Y, K(Y)^*/\mathcal{O}_Y^*) \quad \square.$$

3) follows once we show every invertible $\mathcal{O}_Y$-mod is isom to some invertible sub $\mathcal{O}_Y$-mod of $K(Y)$. Given an invertible $\mathcal{O}_Y$-mod $\mathcal{L}$, we produce an injective $\mathcal{O}_Y$-mod map $\mathcal{L} \to K(Y)$. The image will be the desired invertible $\mathcal{O}_Y$-submod isom to $\mathcal{L}$.

To that end, choose an isom $\mathcal{L}_\eta \xrightarrow[\varphi]{\sim} K(Y)$ where η is the gen pt of Y.

For $U \subseteq_{open} Y$, the canonical map composed with φ

$$\mathcal{L}(U) \longrightarrow \mathcal{L}_\eta \xrightarrow[\varphi]{\sim} K(Y)$$

gives a map of

$$\Psi_U : \mathcal{L}(U) \longrightarrow \mathcal{L}_\eta \xrightarrow{\sim} K(Y)(U) = K(Y)$$

Ψ_U induces a $\mathcal{O}_Y$-lin map $\mathcal{L} \longrightarrow K(Y)$.

Since Y is integral, $\mathcal{L}(U) \hookrightarrow \mathcal{L}_\eta$ and hence Ψ_U is inj $\forall U$. $\square$.

Thm. Assume X is integral and reg in codim 1.

1) The following diagram is commutative

$$\begin{tikzcd}
\mathrm{Cart}(X) \arrow[rr] \arrow[d] \arrow[dr] & & \{\text{subsheaves of } K(X)\} \\
\mathrm{div}(s) & \mathrm{Weil}(X) \arrow[ur] &
\end{tikzcd}$$

$s \longmapsto \mathcal{L}(s)$

$D \longmapsto \mathcal{O}_X(D)$

$s \mapsto \operatorname{div}(s) \longrightarrow \operatorname{Weil}(X) \qquad D$

2) When X is locally factorial, we have a commutative diag of isom

$$\begin{array}{ccc} \operatorname{CaCl}(X) & \longrightarrow & \operatorname{Pic}(X) \\ & \searrow \quad \nearrow & \\ & \operatorname{Weil}(X) & \end{array}$$

Cor: $\mathbb{P}^n_k = \operatorname{Proj}(k[x_0, \dots, x_n])$. Any invertible sheaf on $\mathbb{P}^n_k$ is isom to some $\mathcal{O}(m)$ for some $m \in \mathbb{Z}$.

§. Cartier divisor associated to rational sections of an invertible sheaf.

Let X be an integral scheme, η be the generic pt. $\mathcal{L}$ be an invertible $\mathcal{O}_X$-mod, $0 \neq s \in \mathcal{L}_\eta$. We are going to define $\operatorname{div}(s) \in \operatorname{CaDiv}(X)$.

Choose an open covering $X = \bigcup_{i \in I} U_i$ s.t $\mathcal{L}|_{U_i} \xleftarrow{\sim} \mathcal{O}_{U_i}$ $\forall i \in I$.

For each $i \in I$, fix an isom; let s_i be the image of $1 \in \mathcal{O}_X(U_i)$

Then $s|_{U_i} = f_i \cdot s_i$ for some $f_i \in K(X) - \{0\}$.

Define $\operatorname{div}(s) \in \operatorname{CaDiv}(X)$ by setting
$\operatorname{div}(s) = (U_i, f_i)_{i \in I}$.

Prop/def: $\operatorname{div}(s)$ is independent of the choice $(U_i, s_i \in \mathcal{L}(U_i))$ made above.

Pf: Exercise.

Def. Two Cartier divisors s, s' on X are called linearly equivalent if they represent the same class in $\operatorname{CaCl}(X)$. We write $s \sim s'$ to denote linear equivalence.

Thm: ① For nonzero $s, s' \in \mathcal{L}_\eta$, $\operatorname{div}(s) \sim \operatorname{div}(s')$

② Let $s \in \operatorname{CaDiv}(X)$. There is a bijection

Thm: ① for nonzero $s, s' \in \mathcal{K}_\eta$,

② Let $s \in \mathrm{Cart}(X)$. There is a bijection

$$\{\text{Cartier divisors linearly equivalent to } s\} \longleftrightarrow \underset{\text{Rational sections of } \mathcal{L}(s)}{\mathcal{L}(s)_\eta - \{0\}}.$$

(Check That $1 \in \mathcal{L}(s)$ is mapped to s).

Now assume X is reg in codim 1 and noetherian.
Given an invertible sheaf $\mathcal{L}$ and $0 \neq s \in \mathcal{L}_\eta$, one can consider The Weil divisor corresponding to $\mathrm{div}(s) \in \mathrm{Cart}(X)$, denote it by $\mathrm{div}(\mathrm{div}(s))$

Ex: 1) Realize $\mathrm{div}(\mathrm{div}(s))$ as The divisor of zeros and poles of The rational section.

2) Assume $s \in H^0(X, \mathcal{L})$. Show That The non-vanishing locus we defined D_s, is The same as

$$X - \mathrm{supp}(\mathrm{div}(\mathrm{div}(s))).$$

Where, support of an Weil divisor $D = \sum_{D_i \text{ prime}} a_i D_i$, $a_i \in \mathbb{Z}$, is defined as,
$$\mathrm{supp}(D) = \bigcup_{a_i \neq 0} D_i.$$